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Driven diffusive system with non-local perturbations

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Abstract
We investigate the impact of non-local perturbations on driven diffusive systems. Two different problems are considered here. In one case, we introduce a non-local particle conservation along the direction of the drive and in another case, we incorporate a long-range temporal correlation in the noise present in the equation of motion. The effect of these perturbations on the anisotropy exponent or on the scaling of the two-point correlation function is studied using renormalization group analysis.

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1. Introduction

Driven diffusive systems have a special role in the study of non-equilibrium systems because of the unusual steady-state properties that are not found in equilibrium. The main aspects which are crucial for these new non-equilibrium properties are particle conservation, spatial anisotropy associated with the drive and the existence of a non-equilibrium steady state [1]. A simple paradigmatic model that captures all the above features is the so-called KLS model named after Katz, Lebowitz and Spohn [2, 3]. This model describes diffusing lattice gas particles in contact with a thermal bath. In addition to the attractive interparticle interaction, there is a driving force acting in a specific direction in the d-dimensional space. The rate of hopping is biased along the direction of the drive. The periodic boundary condition along and transverse to the drive implies a toroidal geometry and in this geometry even though the drive looks like a potential locally, there is no global potential. As a result, in the steady state there is a steady flow of particle-current and energy-current looping around the toroid. The steady flow of energy results from the fact that the particles gain energy from the drive and lose it in the thermal bath.

Since the appearance of the KLS model, various techniques such as mean-field theory, renormalization group (RG) analysis, numerical simulations have been employed to investigate its steady state. Although numerical simulations show the presence of the order–disorder transition for all strengths of the drive [4], the behaviour above, below and at criticality is quite different from that in the corresponding undriven case. Below criticality, apart from the critical exponent β or the critical temperature T_c, the most crucial differences from the equilibrium Ising model appear through the shape of the coexistence curve and the shape of the coexisting regions which are strip-like instead of droplets [5]. Above T_c, the most remarkable phenomenon is the existence of power-law correlations [3, 6]. According to the RG ideas in statistical mechanics, such scale invariant properties should be associated with RG fixed points. RG studies on a system of diffusing, non-interacting particles under drive [7] indeed support this idea. This model is argued as a special limit of the KLS model in which the temperature and the drive strength are large with a fixed ratio of the two [8]. The RG analysis shows that for $d > 2$, the high-temperature properties of the KLS model are controlled by a line of fluctuation–dissipation theorem (FDT) violating fixed points in contrast with a single $T = \infty$ fixed point for the equilibrium Ising model. For $d < 2$ there are both FDT restoring and FDT violating locally stable fixed points. Striking differences appear in the universality and scaling in the critical region that has been studied in great detail. The fixed point which governs the critical properties of the undriven system becomes unstable and a new fixed point determines the critical properties of the driven system [9]. As a result, universal exponents are different and, in particular, there is an anisotropy exponent that distinguishes the directions along and perpendicular to the drive. Although exponents are calculated using $\epsilon(=5-d)$-expansion scheme after one-loop RG analysis, they are exact due to the invariance of the system under Galilean transformation.

In this paper, our focus is on the critical region and we are interested in two problems that are related to the particle conservation and the noise in the system (motivations discussed below). In the first problem, we relax the constraint of particle conservation by introducing a non-locality and study its impact on the KLS model. In the second problem, the impact of a long-ranged temporally correlated noise is studied because such temporal correlations are expected to disturb the Galilean invariance.

Particle conservation is an important ingredient in driven diffusive systems. However, particle conservation need not necessarily be maintained locally and it can be satisfied in a more global way. In the lattice-gas language, particles may be interchanged over a long distance unlike the nearest-neighbour spin-exchange Kawasaki dynamics. A few recent studies on critical dynamics of model B [10, 11], model C [12] and model D [13] show that the non-locality in conservation results in drastic changes in the dynamical class. In view of these recent observations and the importance of the conservation in driven diffusive systems, it is natural to ask how crucial is the locality condition of the conservation for the dynamics of the driven diffusive system.

In driven diffusive systems, the noise is usually considered to have a Gaussian distribution with a short-range correlation in space and time. Though a long-range spatial correlation in the noise in the high-temperature version of the KLS model [14] exhibits interesting crossover behaviour [15] with increasing range of the noise correlation, the effect of a long-range temporal correlation near the critical point need not be so. We have already pointed out that several of the known results depend crucially on the Galilean invariance [9] ensuing from the delta-correlation of the noise in time. However, this delta-correlation is not a necessity because the FDT need not be respected in non-equilibrium. Also one notes that any temporally correlated noise would break the Galilean invariance. In fact, long-range temporal correlation in the noise is well known in various problems that include biological, physical and economical systems. Such temporally correlated noise may also occur if the source is another correlated system, which, e.g., can give rise to long-range time correlation at criticality. In our second problem, we study the effect of a temporally correlated noise on the known results of driven diffusive systems described by the KLS model.

Since our focus is on the long distance and large time behaviour of systems, a convenient starting point would be an appropriate continuum model based on the Langevin equation of motion. Our interest is in the critical region where the dynamic RG analysis serves as a useful machinery. We, therefore, adopt a momentum-shell RG technique in the following to address the above questions.

The rest of the paper is organized as follows. In the next section, we discuss the effect of the non-local conservation. After discussing the model and already known results in certain specific cases, we move on to the discussion of our RG results in the next subsection. In section 3, the effect of a temporally correlated noise is discussed. We conclude by summarizing the results in section 4.

2. Non-local particle conservation

The non-locality in the conservation is introduced through a non-local kernel in the particle current density. The local problem which is the usual KLS model can be retrieved by considering the appropriate limit.

2.1. The model and the known results

Associated with the conserved particle density $S(\mathbf{x}, t)$, there is a continuity equation

$$\partial_t S(\mathbf{x}, t) + \partial_{\|} J_{\|} + \nabla_{\perp} \cdot \mathbf{J}_{\perp} = 0 \tag{1}$$

where the $J_{\|}$ and $\mathbf{J}_{\perp}$ are the current densities along and perpendicular to the drive. In the absence of the drive, the deterministic parts of the current densities can be obtained from the gradient of the chemical potential. The total current densities, therefore, have the standard form

$$\mathbf{J}_{\perp} = -\lambda \nabla_{\perp} \frac{\delta \mathcal{H}}{\delta S} + \zeta_{\perp}(\mathbf{x}, t) \tag{2}$$

and

$$J_{\|} = \int \mathrm{d}^d x' \chi(\mathbf{x} - \mathbf{x}') \left[-\lambda \partial'_{\|} \frac{\delta \mathcal{H}}{\delta S(\mathbf{x}')} + \zeta_{\|}(\mathbf{x}', t) \right] \tag{3}$$

where $\zeta_{\perp}$ and $\zeta_{\|}$ are the noisy parts of the current densities perpendicular and parallel to the drive, respectively and λ is the transport coefficient. Here and in the following, $\|$ and $\perp$ subscripts are used to distinguish directions parallel and perpendicular to the drive. In equation (3), $\partial_{\|}$ with a prime acts on the x' coordinate. $\mathcal{H}$ is the usual Ginzburg–Landau Hamiltonian for the lattice gas

$$\mathcal{H} = \int \mathrm{d}^d x \left\{ \frac{1}{2} (\nabla S)^2 + \frac{1}{2} \tau S^2 + \frac{u}{4!} S^4 \right\} \tag{4}$$

where $\tau \propto (T - T_c)$ measures the deviation from the critical temperature. Although the Hamiltonian is isotropic, the drive induces anisotropic τ. This is usually taken care of by introducing two τ, $\tau_{\|}$ and $\tau_{\perp}$, which behave differently as T approaches T_c [16]. We, therefore, need to consider two such τ in our equation of motion. Note that we have incorporated the non-local conservation in the parallel component of the current in equation (3). The local conservation corresponds to $\chi(\mathbf{x} - \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}')$. Since the effect of the drive is expected to be predominant along its direction, we introduce the non-locality only in the parallel direction. Generalization of this, however, is possible along the same line. When the drive is switched on, there is an additional particle current along the drive. This leads to an additive term proportional to $g\partial_{\|} S^2$ in equation (1) with g being related to the strength of the drive. Equations (1)–(4), along with this nonlinear term due to the drive lead to the final Langevin equation which we have written explicitly in the appendix.

Different forms of $\chi(\mathbf{x}-\mathbf{x}')$ can be considered in general. Since any short-range χ under renormalization would look like a delta-function, in the large scale limit, χ is going to matter only if it is long-range. We, therefore, consider a special case where

$$\chi(\mathbf{x}-\mathbf{x}') \sim \delta(\mathbf{x}_\perp - \mathbf{x}'_\perp)(x-x')_\parallel^{\sigma-1}. \tag{5}$$

In particular, in the Fourier space, $\chi(k) = \rho_{nl} k_\parallel^{-\sigma}$ with ρ_{nl} denoting the strength of the non-local conservation. The local conservation corresponds to $\sigma = 0$, and $\sigma = 2$ would imply a global conservation in the parallel direction. Since the inter-particle interaction is expected to remain unaffected by the drive in the transverse direction, we expect $\tau_\perp$ to vanish as the critical temperature is approached. Considering a positive finite value for $\tau_\parallel$, it can be seen that the leading momentum dependent terms in the equation of motion are $k_\perp^4$ and $k_\parallel^{2-\sigma}$. This leads to an anisotropic scaling

$$k_\parallel \sim k_\perp^{1+\Delta} \tag{6}$$

with $\Delta = (2+\sigma)/(2-\sigma)$. Therefore, in the presence of non-local conservation, the anisotropy is more pronounced than the corresponding local case. In the large length scale limit, all the irrelevant terms drop out [1] and in the Fourier space, we have the following equation of motion:

$$\begin{aligned}\left[\mathrm{i}\omega + \lambda k_\perp^4 + \lambda\chi(k)k_\parallel^2\right]S(\mathbf{k},\omega) - \frac{1}{2}\lambda g \mathrm{i} k_\parallel \int \mathrm{d}\mathbf{k}_1\, \mathrm{d}\omega_1 S(\mathbf{k}_1,\omega_1) S(\mathbf{k}-\mathbf{k}_1, \omega-\omega_1) \\ = -\mathrm{i}\mathbf{k}_\perp \cdot \boldsymbol{\zeta}_\perp(\mathbf{k},\omega).\end{aligned} \tag{7}$$

In the Fourier space, the spatially and temporally uncorrelated noise is described by the second moment

$$\langle \mathrm{i}\mathbf{k}_\perp \cdot \boldsymbol{\zeta}_\perp(\mathbf{k},\omega)\mathrm{i}\mathbf{k}'_\perp \cdot \boldsymbol{\zeta}_\perp(\mathbf{k}',\omega')\rangle = 2Dk_\perp^2 \delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega') \tag{8}$$

with D characterizing the noise amplitude. In equation (7), we have dropped a naively irrelevant term $\frac{u}{3!}\nabla_\perp^2 S^3$ which assures stability below criticality. This term, identified as a dangerously irrelevant term, plays a crucial role in determining the equation of state and the critical exponent β for the KLS model [9].

It can be very easily seen from the real-space version of equation (7) that it possesses Galilean invariance [9] under the transformation

$$S(\mathbf{x},t) \to S(\mathbf{x} + \lambda g a \mathbf{e} t, t) + a \tag{9}$$

where a is a small continuous parameter and $\mathbf{e}$ is the unit vector in the direction of the drive. Under this transformation, the stochastic equation (7) is subjected to a noise $\nabla_\perp \cdot \boldsymbol{\zeta}'_\perp(\mathbf{x}',t) = \nabla_\perp \cdot \boldsymbol{\zeta}_\perp(\mathbf{x} + \lambda g a \mathbf{e} t, t)$. The noise correlation is modified as

$$\begin{aligned}&\langle \boldsymbol{\nabla}_\perp \cdot \boldsymbol{\zeta}'_\perp(\mathbf{x}'_1,t_1)\boldsymbol{\nabla}_\perp \cdot \boldsymbol{\zeta}'_\perp(\mathbf{x}'_2,t_2)\rangle \\ &\quad = \langle \boldsymbol{\nabla}_\perp \cdot \boldsymbol{\zeta}_\perp(\mathbf{x}_{1\perp}, x_{1\parallel}\boldsymbol{\zeta}_\perp + \lambda a g t_1, t_1)\boldsymbol{\nabla}_\perp \cdot (\mathbf{x}_{2\perp}, x_{2\parallel} + \lambda a g t_2, t_2)\rangle \\ &\quad = \boldsymbol{\nabla}_\perp^2 F(\mathbf{x}_{1\perp} - \mathbf{x}_{2\perp}, x_{1\parallel} - x_{2\parallel} + \lambda a g(t_1 - t_2), t_1 - t_2)\end{aligned} \tag{10}$$

where, for generality, we have retained an arbitrary function F that represents the noise correlation in the original equation. In the case of a δ-correlation in time, $F(\mathbf{x}_\perp, x_\parallel, t) = \delta(t) f(\mathbf{x}_\perp, x_\parallel)$, the correlation of the new noise is the same as the original one. This, however, is not true when there is a long-range temporal correlation in the noise. Therefore, the stochastic equation (7) is invariant under Galilean transformation only when the noise has a short-range correlation in time.

For local conservation, $\sigma = 0$, a dimensional analysis leads to $\Delta = 1$. Because of the anisotropy, we need to define two ν-like exponents associated with the correlation lengths in the transverse and longitudinal directions. In the same way, one can define two z-like dynamic critical exponents and different η-like anomalous exponents associated with the structure factor. The detailed relationship of these exponents with the original Ising exponents z or η can be worked out [1]. A one-loop RG analysis further reveals that there exists a nontrivial infra-red stable fixed point [9] for the drive that determines the critical behaviour of the system below the upper-critical dimension $d_c = 5$. Universal exponents acquire corrections in the $\epsilon(=5-d)$-expansion scheme and the independent exponents are

$$\Delta = 1 + \frac{\epsilon}{3} \qquad z = 4 \qquad \eta = 0 \qquad \nu = 1/2 \qquad \beta = 1/2. \tag{11}$$

For $d > 5$, the drive is irrelevant and the critical exponents are mean-field like.

2.2. *Renormalization group analysis*

Since the RG analysis in the following does not lead to any renormalization of λ, we set $\lambda = 1$. However, for book-keeping purposes, it is convenient to maintain explicitly a coefficient of $k_\perp^4$ term in the square bracket of equation (7). This coefficient, henceforth, is denoted by $\overline{\nu}$. In the RG analysis, we look for the scale invariance of all the coefficients.

A naive dimensional analysis can be done for the present system. We observe how the equation of motion changes under a change of scale $x_\perp \to bx_\perp$, $x_\parallel \to b^{1+\Delta}x_\parallel$ and $t \to b^z t$, where z is the dynamic exponent. Assuming that the field scales as $S \to b^\chi S$ under this transformation, we find that various parameters scale as

$$\overline{\nu} \to b^{z-4}\overline{\nu} \tag{12}$$

$$\rho_{nl} \to b^{z+(1+\Delta)(\sigma-2)}\rho_{nl} \tag{13}$$

$$g \to b^{\chi-1-\Delta+z} g \tag{14}$$

$$D \to b^{-2-(d+\Delta+z)-2\chi+2z} D. \tag{15}$$

For local conservation, $\sigma = 0$, and for $g = 0$, the invariance of $\overline{\nu}$, ρ_{nl} and D leads to $z = 4$, $\Delta = 1$ and $\chi = (1-d)/2$. These exponents are associated with the trivial fixed point $g = g^* = 0$. Around this fixed point, the drive strength g scales as $b^{(5-d)/2}g$ and this implies that the upper-critical dimension is 5. For $\sigma > 0$, ρ_{nl} is relevant around the trivial fixed point and we perform an RG analysis in the following to study its impact on the known one-loop results.

Many technicalities related to the one-loop dynamic RG analysis done in the following are similar to [17] even though there is no immediate connection between the two problems. We, therefore, skip the details and mention the steps for the sake of completeness. A convenient starting point is to rewrite equation (7) as

$$S(\mathbf{k},\omega) = -\mathrm{i}\mathbf{k}_\perp \cdot \boldsymbol{\zeta}_\perp(\mathbf{k},\omega)G_0(\mathbf{k},\omega) + \frac{1}{2}gG_0(\mathbf{k},\omega)(\mathrm{i}k_\parallel)\int \mathrm{d}\mathbf{k}_1\,\mathrm{d}\omega_1\,S(\mathbf{k}_1,\omega_1)S(\mathbf{k}-\mathbf{k}_1,\omega-\omega_1) \tag{16}$$

with a bare propagator

$$G_0(\mathbf{k},\omega) = \left(\mathrm{i}\omega + \overline{\nu}k_\perp^4 + \rho_{nl}k_\parallel^{2-\sigma}\right)^{-1}. \tag{17}$$

The bare vertex is given by $\frac{1}{2}g(\mathrm{i}k_\parallel)$. Diagrammatic representations of this equation and the vertex are shown in figure 1.

The normal procedure is to perform an iterative expansion where S in equation (16) is replaced by itself. This is continued up to the desired order and different terms in the series are, then, averaged over the noise. Various terms in the perturbative series, thus obtained, give

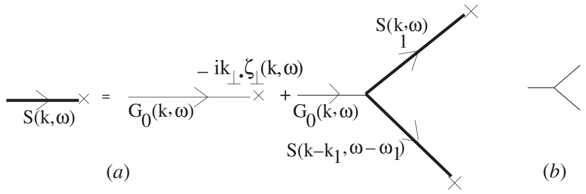

Figure 1. (*a*) Diagrammatic representation of equation (16). (*b*) Three-point vertex. The symbol (×) represents the noise and a thin line with an arrow represents a bare propagator.

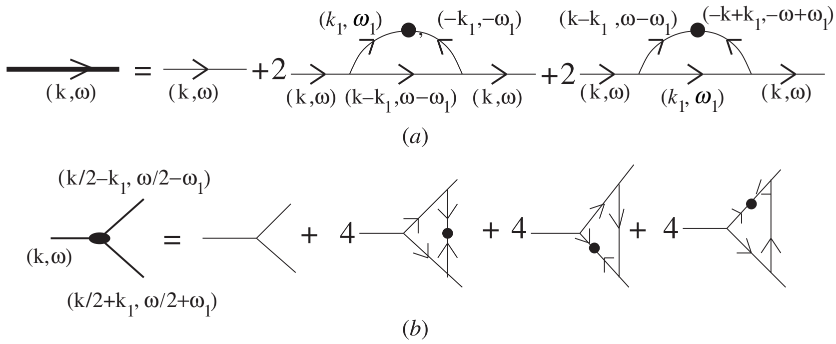

Figure 2. (*a*) Diagrams contributing to the renormalization of the propagator. (*b*) Diagrams contributing to the renormalization of the vertex. On the left-hand sides, we have the effective propagator (in (*a*)) and the effective vertex (in (*b*)). Filled dots represent noise contraction. Numbers in front of the diagrams are the combinatorial factors that originate from possible noise contractions.

rise to effective average quantities. We find that the effective propagator $G(\mathbf{k},\omega)$ defined by $S(\mathbf{k},\omega) = -\mathrm{i}\mathbf{k}_\perp \cdot \boldsymbol{\zeta}(\mathbf{k},\omega)G(\mathbf{k},\omega)$ is

$$G(\mathbf{k},\omega) = G_0(\mathbf{k},\omega) - 2\left(\tfrac{1}{2}g\right)^2 2Dk_z G_0^2(\mathbf{k},\omega)\left\{\int \frac{\mathrm{d}\mathbf{k}_1\,\mathrm{d}\omega_1}{(2\pi)^{d+1}}\left(k_{1z}+\tfrac{1}{2}k_z\right)(\mathbf{k}/2-\mathbf{k}_1)_\perp^2 \right.$$
$$\times\, G_0(\mathbf{k}_1+\mathbf{k}/2,\omega_1+\omega/2)\,|G_0(\mathbf{k}_1-\mathbf{k}/2,\omega_1-\omega/2)|^2$$
$$+\int \frac{\mathrm{d}\mathbf{k}_1\,\mathrm{d}\omega_1}{(2\pi)^{d+1}}(k_z/2-k_{1z})(\mathbf{k}_1+\mathbf{k}/2)_\perp^2 G_0(\mathbf{k}/2-\mathbf{k}_1,\omega/2-\omega_1)$$
$$\left.\times\, |G_0(\mathbf{k}_1+\mathbf{k}/2,\omega_1+\omega/2)|^2\right\}. \qquad (18)$$

A diagrammatic representation of the terms appearing in this equation is given in figure 2(*a*). The one-loop corrections obtained from the perturbative series have divergences in the $k_1 \to 0$ limit. The renormalization group amounts to avoiding these singularities by re-summation. There is also an ultra-violet cutoff Λ in the momentum and this is due to the existence of a short-distance cutoff in the system. We use the momentum shell renormalization scheme, where fluctuations over the momentum shell $\Lambda\,\mathrm{e}^{-l} < k_1 < \Lambda$ are integrated out and subsequently a rescaling in the momentum as $k_1 \to k_1\,\mathrm{e}^{-l}$ is done to retrieve the original cutoff Λ. The rescaling of momentum is the same as that carried out earlier with $b = \mathrm{e}^l$.

An effective noise amplitude $\tilde{D}$ can be defined in a similar way as

$$\langle S^*(\mathbf{k},\omega)S(\mathbf{k},\omega)\rangle = 2k_\perp^2\tilde{D}G(\mathbf{k},\omega)G(-\mathbf{k},\omega). \qquad (19)$$

To see the renormalization of the drive, one has to calculate the effective three-point vertex function Γ, where the bare vertex function is given by $\Gamma_0 = \mathrm{i}k_\parallel(g/2)$. However, it can be checked that no one-loop correction to the effective noise amplitude is present here. This is essentially due to the fact that the nonlinear term due to the drive is accompanied by an overall factor $k_\parallel$. The drive term also does not require any renormalization and this is due to the Galilean invariance which is valid in this case as well. RG transformation, being analytic in nature, cannot renormalize ρ_{nl}.

It is easy to check that starting with the propagator in equation (17), a term proportional to $k_\parallel^2$ is always generated. We, therefore, incorporate this term, with a coefficient ρ_l, in the bare propagator $G_0(\mathbf{k}, \omega)$ which now appears as

$$G_0(\mathbf{k}, \omega) = \left(\mathrm{i}\omega + \overline{\nu}k_\perp^4 + \rho_{nl}k_\parallel^{2-\sigma} + \rho_l k_\parallel^2\right)^{-1}. \tag{20}$$

In the hydrodynamic limit, $(\mathbf{k}, \omega) \to 0$, the effective ρ_l is given by

$$\rho_l^{\mathrm{eff}} = \rho_l^0 + (g/2)^2[A + B] \tag{21}$$

where

$$A = \int \frac{\mathrm{d}\mathbf{k}_1\,\mathrm{d}\omega_1}{(2\pi)^{d+1}} k_{1\perp}^2 G_0(-k_1, -\omega_1)^2 G_0(k_1, \omega_1) \tag{22}$$

and

$$B = \int \frac{\mathrm{d}\mathbf{k}_1\,\mathrm{d}\omega_1}{(2\pi)^{d+1}} k_{1\perp}^2 G_0(-k_1, -\omega_1) G_0(k_1, \omega_1)^2. \tag{23}$$

To do the momentum shell integration, we take care of the anisotropy in the transverse and longitudinal directions by choosing polar coordinates [18]

$$\overline{\nu}^{1/2}k_{1\perp}^2 = r\sin\theta \qquad \rho_l k_{1\parallel}^2 = r^2\cos^2\theta. \tag{24}$$

The momentum integration over the shell with inner and outer radii $\Lambda\,\mathrm{e}^{-l}$ and Λ with infinitesimal l leads to

$$\rho_l^{\mathrm{eff}} = \rho_l^o + \frac{g^2 D}{8{\rho^o_l}^{1/2}\overline{\nu}^{(d+1)/4}} l \frac{S_{d-1}}{(2\pi)^d} I(d, \sigma, \rho_r) \tag{25}$$

where

$$I(d, \sigma, \rho_r) = \int \mathrm{d}\theta \frac{\sin^{(d-1)/2}\theta}{(1 + \rho_r\cos^{2-\sigma}\theta)^2} \tag{26}$$

and

$$\rho_r = \rho_{nl}/\rho_l^{(2-\sigma)/2}. \tag{27}$$

S_{d-1} represents the surface area of a unit $d-1$ dimensional sphere. In equation (25), we have used $\Lambda = 1$.

After rescaling to retrieve the original momentum cutoff, we have the following RG equations, describing the flow of the parameters under the change of the length scale

$$\frac{\mathrm{d}\overline{\nu}}{\mathrm{d}l} = (z-4)\overline{\nu} \tag{28}$$

$$\frac{\mathrm{d}\rho_{nl}}{\mathrm{d}l} = [z - (2-\sigma)(1+\Delta)]\rho_{nl} \tag{29}$$

$$\frac{\mathrm{d}\rho_l}{\mathrm{d}l} = \rho_l\left[(z - 2(1+\Delta)) + \frac{uD}{8}\frac{K_{d-1}}{2\pi} I(d, \sigma, \rho_r)\right] \tag{30}$$

$$\frac{\mathrm{d}D}{\mathrm{d}l} = (z - 2\chi - d - 2 - \Delta)D \tag{31}$$

$$\frac{\mathrm{d}u}{\mathrm{d}l} = \left[2\chi + (1+\Delta) + z/2 - \frac{1}{4}(d+1)(z-4)\right]u - \frac{3}{16}u^2 D \frac{K_{d-1}}{2\pi} I(d, \sigma, \rho_r) \tag{32}$$

where $u = g^2 \rho_l^{-3/2} \overline{\nu}^{-\frac{1}{4}(d+1)}$ and $K_{d-1} = S_{d-1}/(2\pi)^{d-1}$.

An RG equation for ρ_r can be obtained from equations (29) and (30),

$$\frac{\mathrm{d}\rho_r}{\mathrm{d}l} = \rho_r \left[\frac{\sigma z}{2} - (2-\sigma)\frac{uD}{16}\frac{S_{d-1}}{(2\pi)^d} I(d, \sigma, \rho_r)\right]. \tag{33}$$

The trivial fixed point $\rho_r^* = 0$ corresponds to the local conservation. In that case, there exists a non-Gaussian infra-red stable fixed point $u^* = 32\epsilon/3C_1$ where $C_1 = \frac{S_{d-1}}{(2\pi)^d}\sqrt{\pi}\,\Gamma[(1+d)/4]/\,\Gamma[(3+d)/4]$. Further, the scale invariance of $\overline{\nu}$ and ρ_l implies $z = 4$ and $\Delta = 1 + \epsilon/3$ respectively. The invariance of D under RG transformation leads to $\chi = (2 - d - \Delta)/2$. Substituting the expression of Δ, we have $\chi = \frac{1}{2}(1 - d - \epsilon/3)$. The exponent χ describes the scaling of the correlation function $\langle S(\boldsymbol{x}, t) S(\mathbf{0}, 0)\rangle$. In the presence of the anisotropic scaling, this correlation function scales as

$$\langle S(\boldsymbol{x}, t) S(\mathbf{0}, 0)\rangle \sim x_\perp^{2\chi} f_1\big(x_\parallel / x_\perp^{1+\Delta}, t/x_\perp^{z}\big) \tag{34}$$

where f_1 is an appropriate scaling function. A nontrivial fixed point for ρ_r exists if

$$\frac{\sigma z}{2} = (2-\sigma)\frac{uD}{16}\frac{K_{d-1}}{2\pi} I(d, \sigma, \rho_r). \tag{35}$$

Since the nontrivial fixed point for u is

$$u^* = (3z/2 - 1 - d)\frac{16}{3DI(d, \sigma, \rho_r) K_{d-1}/(2\pi)} \tag{36}$$

the exponent relation

$$\frac{\sigma z}{2} = \frac{(2-\sigma)}{3}(3z/2 - 1 - d) \tag{37}$$

is satisfied whenever $u = u^*$ and $\mathrm{d}\rho_r/\mathrm{d}l = 0$. With $z = 4$, one finds a critical value of σ, $\sigma_\mathrm{c} = \frac{2\epsilon}{\epsilon+6}$ from equation (37). At this critical value, ρ_r is marginal and the scale invariance of ρ_{nl} in equation (29) yields $\Delta = 1 + \epsilon/3$. Obviously, this value of Δ also leads to the invariance of ρ_l. For $\sigma < \sigma_\mathrm{c}$, ρ_r is irrelevant and flows to zero in the asymptotic limit. The fixed point in equation (36) and the anisotropy exponent are the same as those in the local conservation case. For $\sigma > \sigma_\mathrm{c}$, ρ_r is relevant and one needs to extend equation (33) beyond one loop to get a fixed point. If there exists a fixed point, invariance of ρ_{nl} would lead to $\Delta = (2+\sigma)/(2-\sigma)$. A numerical simulation is likely to reveal the universal properties for large σ.

It may be noted that in many cases the approach to the short-range limit is somewhat subtle. For ferromagnets with long-range exchange of the form $r^{-(d+\sigma_\mathrm{e})}$, with r as the distance between two spins, critical exponents show apparent discontinuity as the short-range limit $\sigma_\mathrm{e} \to 2$ [19] is approached. Later, this problem was sorted out by Sak [20], in his momentum-shell RG analysis, by taking the corresponding local term that is generated under renormalization. Although irrelevant, this local term becomes at least marginal as $\sigma_\mathrm{e} \to 2$ and competes with the non-local term. This interplay leads to the fact that as σ_e approaches 2 from below, short-range critical exponents are found if $\sigma_\mathrm{e} > \sigma_\mathrm{ec}$ where σ_ec is determined by short-range exponents. In the field theoretic RG formulation, the apparent discontinuity is removed by performing a double expansion in the corresponding $\epsilon (= d_\mathrm{c} - d)$ and the deviation of σ_e from its short-range value [21]. As the short-range limit is approached, this deviation is small and is treated on an equal footing with ϵ. The problem of a driven diffusive system with a long-range spatially correlated noise [15] is more intricate due to the existence of two fixed points in the short-range case and only one fixed point in the corresponding long-range version.

In our case, the discontinuity-related problem is naturally taken care of by the local term that is generated. In a similar way as [20], the short-range behaviour is retrieved for $\sigma < \sigma_c$, where σ_c is obtained from the short-range exponents.

A more general form of the non-local conservation can exhibit what happens in the case of completely global conservation. This generalization requires a reconsideration of the scaling and relevance or irrelevance of all the terms originally present in the model. In our case, for the global conservation limit, $\sigma \to 2$, one would require such consideration since there is no $k_\parallel$ dependent term in the free part of equation (7).

3. Temporally correlated noise

In this section, we consider a noise with a long-range temporal correlation. A large simplification in the RG analysis follows from the invariance of the system under Galilean transformation described in equation (9). The RG transformation, that preserves this symmetry, assures that the drive is not renormalized in the process. With a long-range temporal correlation in the noise, the Galilean invariance is lost and the flow equation associated with the drive acquires contributions from one-loop terms in the perturbation series. To obtain this, we, here, proceed with the noise correlation

$$\langle i\mathbf{k}_\perp \cdot \zeta_\perp(k,\omega) i\mathbf{k}'_\perp \cdot \zeta_\perp(k',\omega')\rangle = 2Dk_\perp^2 \omega^{-2\theta_d}\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega'). \tag{38}$$

For $\theta_d = 0$, all the known results are expected to follow.

The renormalization of ρ_l follows from the same expansion done for equation (16), except for a new factor $|\omega|^{-2\theta_d}$ associated with the averaging over the noise in the present case. The three-point vertex function now acquires contributions from one-loop terms. Three diagrams that contribute to the renormalized vertex at the one-loop level are shown in figure 2(*b*). Evaluating these diagrams, one finds the effective drive

$$g_{\text{eff}} = g\Bigg[1 + 4\left(\frac{1}{2}g\right)^2 2D\int \frac{\mathrm{d}\mathbf{k}_1\,\mathrm{d}\omega_1}{(2\pi)^{d+1}} k_{1\parallel}^2 k_{1\perp}^2 G_0(\mathbf{k}_1,\omega_1)^2 G(-\mathbf{k}_1,-\omega_1)^2|\omega_1|^{-2\theta_d} - 8\left(\frac{1}{2}g\right)^2 2D\int \frac{\mathrm{d}\mathbf{k}_1\,\mathrm{d}\omega_1}{(2\pi)^{d+1}} k_{1\parallel}^2 k_{1\perp}^2 |\omega_1|^{-2\theta_d} G_0(\mathbf{k}_1,\omega_1)^2 |G(\mathbf{k}_1,\omega_1)|^2\Bigg]. \tag{39}$$

After a straightforward evaluation of the one-loop contribution over the momentum shell and subsequent rescaling of the momentum, we obtain the flow equation for the drive

$$\frac{\mathrm{d}g}{\mathrm{d}l} = g\left\{(z+\chi-1-\Delta) - \frac{K_{d-1}}{2\pi} g^2 D\rho_l^{-3/2}\overline{\nu}^{-1/4(d+1)} B\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{(2\pi)}\frac{(1-3\omega^2)}{(1+\omega^2)^3}|\omega|^{-2\theta_d}\right\} \tag{40}$$

where $B = \int_0^{\pi/2} \mathrm{d}\theta \sin^{(d-1)/2}\theta\cos^2\theta = \frac{\sqrt{\pi}}{4}\Gamma[(1+d)/4]/\Gamma[(7+d)/4]$. As expected, for $\theta_d = 0$, the above integral over ω vanishes. In terms of u, we have the RG flow equations

$$\frac{\mathrm{d}\overline{\nu}}{\mathrm{d}l} = \overline{\nu}(z-4) \tag{41}$$

$$\frac{\mathrm{d}D}{\mathrm{d}l} = [z - 2\chi - 2 - d - \Delta + 2z\theta_d]D \tag{42}$$

$$\frac{\mathrm{d}\rho_l}{\mathrm{d}l} = \rho_l\left[(z-2-2\Delta) + u\frac{D}{8}\frac{K_{d-1}}{2\pi}A(1+2\theta_d)\sec(\pi\theta_d)\right] \tag{43}$$

$$\frac{\mathrm{d}u}{\mathrm{d}l} = u\left[\frac{3z}{2} + 2z\theta_d - (d+1) - \frac{1}{4}(d+1)(z-4)\right]$$
$$- u^2 A\left(\theta_d B/A - \frac{3}{16}\right) D\frac{K_{d-1}}{2\pi}(1+2\theta_d)\sec\pi\theta_d \tag{44}$$

where $A = \int_0^{\pi/2} \mathrm{d}\theta \sin^{(d-1)/2}\theta = \frac{\sqrt{\pi}}{2}\Gamma[(1+d)/4]/\Gamma[(3+d)/4]$. As before, the noise amplitude D remains un-renormalized here. In the case of short-range temporal correlation, the only quantity that requires renormalization is ρ_l. Although the drive is not renormalized there due to the Galilean invariance, the effective drive dependent parameter u acquires a one-loop like term as a consequence of its definition. As a result of this interplay between ρ_l and u and the Galilean invariance, the anisotropy exponent becomes exact. In our case, the drive is renormalized independently and, therefore, one-loop results are not exact anymore.

Scale invariance of $\overline{\nu}$ and D leads to

$$z = 4 \qquad \text{and} \quad \chi = \tfrac{1}{2}(2 - d - \Delta + 8\theta_d). \tag{45}$$

A θ_d-dependent fixed point for u follows from equation (44). At this fixed point

$$\frac{\mathrm{d}\rho_l}{\mathrm{d}l} = \rho_l\left[z - 2 - 2\Delta - \frac{\epsilon_\theta}{8}\frac{1}{(\theta_d B/A - 3/16)}\right] \tag{46}$$

where $\epsilon_\theta = 5 - d + 8\theta_d$. The scale invariance of ρ_l now leads to a θ_d-dependent anisotropy exponent

$$\Delta = 1 - \frac{\epsilon_\theta}{16(2\theta_d/(3+d) - 3/16)}. \tag{47}$$

Using this value of the anisotropy exponent at the nontrivial fixed point, we have a θ_d-dependent χ that describes the scaling of the correlation function in equation (34) in the presence of a temporally correlated noise. All the universal exponents go over to the corresponding short-range values continuously as $\theta_d \to 0$.

4. Conclusion

To summarize our results, we have studied the effect of weakening of two basic ingredients, namely the particle conservation and the Galilean invariance, responsible for nontrivial scaling in driven diffusive systems. We have done this by introducing a non-locality in the conservation of particles and a temporally correlated noise. With a long-range temporal correlation of the noise, the exactness of the previously known one-loop results is destroyed due to the breakdown of the Galilean invariance. We find a new anisotropic exponent that depends on the power of the long-range correlation. The scaling of the two-point correlation with the change of the length scale is also found. For a non-local particle conservation, we show how the one-loop RG calculation is modified by considering a simple modification of the KLS model. The non-locality is introduced through a non-local kernel in the parallel component of the Langevin current. This procedure can be extended to study the effect of more general forms of non-local conservation. Our dynamic RG analysis shows that the non-locality is irrelevant below a critical power σ_c and is marginal for $\sigma = \sigma_\mathrm{c}$. Above σ_c, the non-locality is relevant. The relevance of the non-locality deserves a more detailed investigation since there exists a possibility of a crossover to a regime with different universal exponents. Besides the theoretical interest, non-local dynamics has its own relevance due to speeding up of numerical simulations implemented by different kinds of non-local moves [22]. We expect that incorporating various non-local features in the simulations of driven systems would be interesting and also would be a crucial check of the theory employed here.

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Appendix. The full Langevin equation

The full Langevin equation that follows from equations (1)–(4) in a straightforward way is

$$\begin{aligned}\partial_t S(\mathbf{x},t) = {} & \lambda\nabla_\perp^2\left(\tau_\perp S - \nabla_\perp^2 S\right) + \lambda\frac{u}{3!}\nabla_\perp^2 S^3 + \lambda\partial_\parallel \int \mathrm{d}^d x'\, \chi(\mathbf{x}-\mathbf{x}')\partial_\parallel'\left(\tau_\parallel S' - \partial_\parallel'^2 S'\right) \\ & + \lambda\kappa\frac{u}{3!}\partial_\parallel \int \mathrm{d}^d x'\, \chi(\mathbf{x}-\mathbf{x}')\partial_\parallel' S'^3 - \lambda\nabla_\perp^2\partial_\parallel^2 S \\ & - \lambda\partial_\parallel \int \mathrm{d}^d x'\, \chi(\mathbf{x}-\mathbf{x}')\partial_\parallel'(\nabla'^2 S') + g\partial_\parallel S^2 \\ & - \nabla_\perp\cdot\boldsymbol{\zeta}_\perp(\mathbf{x}_\perp,t) - \partial_\parallel \int \mathrm{d}^d x'\, \chi(\mathbf{x}-\mathbf{x}')\zeta_\parallel(\mathbf{x}',t). \end{aligned} \tag{A1}$$

Unlike S, S' is a function of the $\mathbf{x}'$ coordinate. A scaling analysis similar to that done for the usual KLS model ($\sigma = 0$) near criticality [1] implies that terms that are irrelevant in the local case remain irrelevant in the presence of a more pronounced anistropy for the non-local case ($\sigma \neq 0$).

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